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Nonlinear Vibrations of Fractionally Damped Systems

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Abstract. This paper deals with the harmonic oscillations of periodically excited nonlinear systems where hysteresis is simulated via fractional operator representations. Employing a diophantine version of the fractional operational powers, the energy constrained Lindstedt–Poincaré perturbation procedure is utilized to establish the harmonic solution. The constrained perturbation procedure was employed since it allows for the handling of strong damping and exciting forces over the full span of the driving frequency range. Based on the approach taken, the long time behavior of the fractionally damped Duffing's equation is studied in detail. Of special interest is the determination of the influence of fractional order on the frequency amplitude response behavior.

Keywords: Fractionally damped, nonlinear systems, Duffing's equation, perturbation procedure.

1. Introduction

The mechanical vibrations of typical engineering systems are influenced by kinematic, kinetic (inertial) as well as hysteretic problem features. While the effects of kinematic and inertial characteristics have been studied in a literal multitude of investigations [1–3], significantly less is available which deals with dissipative behavior. This is especially true of nonlinear systems. The modeling of hysteresis is generally undertaken via either integral or differential formulations [4–6]. Of the two, the differential, e.g., Maxwell–Kelvin–Voigt scheme is most often employed [4]. This is partially a result of the relative ease of fitting [4] as well as of the sensitivity of the integral form to violate thermodynamics, e.g., thermodynamic considerations [7]. The main difficulty of differential formulations lies in their potential numerical stiffness arising from the need to employ many higher order terms to simulate potential frequency dependent hysteretic behavior [7, 8]. As an alternative, fractional differ-integro operators can be employed to represent system hysteresis [8–12]. Excellent monographs are available which define the overall operational properties of such representations [13, 14]. While the fractional differ-integro simulation presents the user with an interpretational dilemma, recent applications in viscoelasticity and structural dynamics [8–12] have pointed to superior fitting characteristics.

To date no work is available regarding the influence of fractional representations on the response behavior of nonlinear systems. In this context, this paper will investigate such effects. Special emphasis will be given to the steady state harmonic response for Duffing type [1, 15] nonlinearity under the influence of periodic inputs with wide ranging frequencies. For the current purposes, a diophantine [8, 9] type fraction simulation will be employed. This enables the problem to be reduced to the fractional equivalent of a first order vector form, e.g., a

$(1/m)$th operator form, wherein m is an integer. From this point, a constrained Lindstedt–Poincaré perturbation scheme [15] is employed to set up the asymptotic small parameter solution. An integral constraint is employed to wash out sub/super harmonic effects [15]. Of special interest is a determination of the influence of fractional order on the amplitude frequency response behavior.

In the sections which follow, detailed discussions will be given on:

1. the definition of the problem,
2. the reduction to $(1/m)$th operator form,
3. the constrained perturbation solution, and
4. the functional effects of fractionally defined hysteresis on system response behavior, e.g., amplitude-frequency response as a function of fractional order.

2. Governing Equations – $(1/m)$th Order Form

For a fractionally damped representation, the Duffing type equation takes the form:

$$MD_2(x) + \sum_{i=1}^{I} C_i D_{q_i}(x) + Kx + \mu(x)^3 = f(t), \tag{1}$$

where M, K, f, μ, I, C_i and q_i are respectively the mass, linear stiffness, the exciting force, a small parameter, an integer defining the number of fractional operators in the simulation, fitting coefficients, and lastly the family of operator powers. The fractional operator $D_{q_i}(\cdot)$ is defined by the expression [13]

$$D_{q_i}(x) = \frac{1}{\Gamma(-q_i)} \int_0^t \frac{1}{(t-\tau)^{q_i+1}}\, x(\tau)\, \mathrm{d}\tau, \tag{2}$$

where Γ is the gamma function. Generally, q_i represent arbitrary irrational numbers [13]. Depending on the accuracy required, these can usually be approximated via diophantine simulations [8], e.g.,

$$q_i = \frac{i}{m}\ , \tag{3}$$

such that m and i are integers. Under the proper choice of i and m, essentially any irrational number can be defined. Based on Equation (3), Equation (2) reduces to

$$D_{q_i}(\cdot) \equiv D_{i/m}(\cdot). \tag{4}$$

To take advantage of Equation (4), a general decomposition rule for differintegration is introduced, namely [8, 13]

$$D_\alpha(D_\beta(R)) = D_{\alpha+\beta}(R) - D_{\alpha+\beta}(R - D_{-\beta}(D_\beta(R))), \tag{5}$$

where α and β are fractions. For large times, it can be shown [8] that

$$D_{\alpha+\beta}(R) \equiv D_\alpha(D_\beta(R)). \tag{6}$$

In the context of Equation (6), Equation (4) can be recast in the form

$$D_{q_i}(\cdot) = D_{i/m}(\cdot) = \prod_{\ell=1}^{i} D_{1/m}(\cdot) \qquad D_{1/m} D_{1/m} \ldots D_{1/m}(\cdot), \tag{7}$$

where $m \neq 0$. Similarly, it follows that

$$D_2(\cdot) = D_{2m/m}(\cdot) = \prod_{\ell=1}^{2m} D_{1/m}(\cdot) = D_{1/m} D_{1/m} \ldots D_{1/m}(\cdot). \tag{8}$$

Introducing Equations (7, 8) into Equation (1) yields

$$M \prod_{\ell=1}^{2m} D_{1/m}(x) + \sum_{i=1}^{I} C_i \prod_{\ell=1}^{i} D_{1/m}(x) + Kx + \mu(x)^3 = f(t). \tag{9}$$

Dividing both sides of Equation (9) by M we arrive at the following form

$$\prod_{\ell=1}^{2m} D_{1/m}(x) + \sum_{i=1}^{I} C_{Mi} \prod_{\ell=1}^{i} D_{1/m}(x) + K_M x + \varepsilon(x)^3 = f_M(t), \tag{10}$$

where $C_{Mi} = C_i/M$, $K_M = K/M$, $f_M = f/M$, and $\varepsilon = \mu/M$.

Now let us define the following transformations

$$\begin{aligned} x &= Y_0, \\ D_{1/m}(Y_0) &= Y_1, \\ D_{1/m}(Y_1) &= Y_2, \\ &\vdots \\ D_{1/m}(Y_\ell) &= Y_{\ell+1}. \end{aligned} \tag{11}$$

Applying Equation (11) to (10) yields that

$$\prod_{\ell=1}^{2m} D_{1/m}(x) = \begin{cases} D_{1/m}(Y_{2m-1}); & \text{for} \quad 2m > I, \\ Y_{2m}; & \text{for} \quad 2m < I, \end{cases} \tag{12}$$

and

$$\prod_{\ell=1}^{i} D_{1/m}(x) = \begin{cases} Y_i; & \text{for} \quad i < I, \\ Y_i; & \text{for} \quad i = I \quad \text{and} \quad I < 2m, \\ D_{1/m}(Y_{I-1}); & \text{for} \quad i = I \quad \text{and} \quad I > 2m. \end{cases} \tag{13}$$

Therefore, the following two forms are possible for Equation (10), namely

(a) For $2m > I$:

$$D_{1/m}(Y_{2m-1}) + \sum_{i=1}^{I} C_{Mi} Y_i + K_M Y_0 + \varepsilon(Y_0)^3 = f_M(t). \tag{14}$$

(b) For $2m < I$:

$$Y_{2m} + C_{Mi} D_{1/m}(Y_{I-1}) + \sum_{i=1}^{I-1} C_{Mi} Y_i + K_M Y_0 + \varepsilon (Y_0)^3 = f_M(t). \tag{15}$$

Using Equation (11), Equations (14) and Equation (15) can be represented in a vector form:

(a) For $2m > I$:

$$D_{1/m}(\underline{Y}) = A_{M0}\underline{Y} + \varepsilon A_{M1}\ \mathrm{Diag}(\underline{Y})\ \mathrm{Diag}(\underline{Y})\underline{Y} + \underline{f}_M(t), \tag{16}$$

where $\underline{Y} = (Y_0 \ \ Y_1 \ \ \ldots \ \ Y_{2m-1})^T$, $\underline{f}_M = (0 \ \ 0 \ \ \ldots \ \ 0 \ \ f_M)^T$, and

$$A_{M0} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & & 0 & 0 \quad\ \ 0 \\ 0 & 0 & 0 & & 0 & 0 \quad\ \ 0 \\ \vdots & & & \ddots & \vdots & \qquad \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots 0 \\ \vdots & & & & \vdots & \qquad \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots 1 \\ -K_M & -C_{M1} & -C_{M2} & \cdots & -C_{MI} & 0 \cdots 0 \end{bmatrix}, \tag{17}$$

$$A_{M1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{18}$$

(b) For $2m < I$; Equation (16) stays as is, but A_{M0}, A_{M1}, and $\underline{f}_M$ change, namely

$$A_{M0} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 & & 0 \\ 0 & 0 & 0 & & 0 & & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ -\dfrac{K_M}{C_{MI}} & -\dfrac{C_{M1}}{C_{MI}} & -\dfrac{C_{M2}}{C_{MI}} & \cdots & -\dfrac{(C_{M2m}+1)}{C_{MI}} & \cdots & -\dfrac{C_{MI-1}}{C_{MI}} \end{bmatrix}, \tag{19}$$

$$A_{M1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -\dfrac{1}{C_{MI}} & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{20}$$

and $\underline{f}_M = (0 \;\; 0 \;\; \ldots \;\; 0 \;\; f_M/C_{MI})^T$, with associated $\underline{Y} = (Y_0 \;\; Y_1 \;\; \ldots \;\; Y_{I-1})^T$.

Equation (16) represents a $(1/m)$th order nonlinear ordinary differential equation. Had Equation (1) been a system of equations, then Equation (16) can be directly extended to matrix form by considering $K_M, C_{MI}, \ldots, \underline{Y}$ and so on as matrix/vector partitions.

3. Constrained Perturbation Solution

The constrained perturbation procedure was employed since it allows for the handling of strong damping and exciting forces over the full span of the driving frequency range. For periodic/harmonic forcing functions, it follows that

$$f(t) = f(t + nP_T), \tag{21}$$

where P_T is the period and n is an integer. Given trigonometric type behavior, then $f(t)$ can be expressed as

$$f(t) = f \cos(\omega t). \tag{22}$$

Hence $\underline{f}_M$ reduces to

$$\underline{f}_M = \underline{F}_M \cos(\omega t). \tag{23}$$

Based on Equation (23), it follows that $\underline{Y}$ takes the form

$$\underline{Y} = \underline{Y}(\omega t). \tag{24}$$

In terms of Equation (24), it can be shown that

$$D_{1/m}(\underline{Y}) = \frac{1}{\Gamma(-1/m)} \int_0^t \frac{1}{(t-\xi)^{1/m+1}} \underline{Y}(\omega\xi) \, \mathrm{d}\xi. \tag{25}$$

To establish a secular term free solution, the Lindstedt–Poincaré time shift is introduced, namely [1, 15]:

$$\tau = \omega t. \tag{26}$$

Employing Equation (26), for the large time state, $D_{1/m}(\underline{Y})$ reduces to

$$D_{1/m}(\underline{Y}(t)) = \frac{1}{\Gamma(-1/m)} \int_0^\tau \frac{1}{(t-\eta/\omega)^{1/m+1}} \underline{Y}(\eta) \frac{\mathrm{d}\eta}{\omega}\,; \tag{27}$$

hence,

$$D_{1/m}(\underline{Y}(t)) \approx \omega^{1/m} D_{1/m}(\underline{Y}(\tau)). \tag{28}$$

Employing Equation (28), Equation (16) reduces to

$$\omega^{1/m} D_{1/m}(\underline{Y}) = A_{M0}\underline{Y} + \varepsilon A_{M1}\,\mathrm{Diag}(\underline{Y})\,\mathrm{Diag}(\underline{Y})\underline{Y} + \underline{F}_M \cos(\tau). \tag{29}$$

To proceed further, let

$$\omega^{1/m} = \Omega \tag{30}$$

and expand $(\Omega, \underline{Y})$ in perturbation series, e.g.,

$$\Omega = \sum_{\ell=0}^{\infty} (\varepsilon)^{\ell} \Omega_{\ell}, \quad \underline{Y} = \sum_{\ell=0}^{\infty} (\varepsilon)^{\ell} \underline{Y}_{\ell}. \tag{31}$$

Therefore, it follows that Equation (29) yields the following perturbational hierarchy of fractional DEQ, namely

$$\Omega_0 D_{1/m}(\underline{Y}_0) = A_{M0}\underline{Y}_0 + \underline{F}_M \cos(\tau), \tag{32}$$

$$\Omega_0 D_{1/m}(\underline{Y}_1) + \Omega_1 D_{1/m}(\underline{Y}_0) = A_{M0}\underline{Y}_1 + A_{M1}\,\mathrm{Diag}(\underline{Y}_0)\,\mathrm{Diag}(\underline{Y}_0)\underline{Y}_0, \tag{33}$$

$$\begin{aligned} &\Omega_0 D_{1/m}(\underline{Y}_2) + \Omega_1 D_{1/m}(\underline{Y}_1) + \Omega_2 D_{1/m}(\underline{Y}_0) \\ &\quad = A_{M0}\underline{Y}_2 + 3A_{M1}\,\mathrm{Diag}(\underline{Y}_1)\,\mathrm{Diag}(\underline{Y}_0)\underline{Y}_0, \end{aligned} \tag{34}$$

and so on.

To constrain the choice of the hierarchy of Ω_i; $i \in [1, \infty]$, we shall use the 'linearized' (e.g., contributed by linear terms) potential energy (PE) accumulated by the system over one cycle, e.g.,

$$\mathrm{PE} = \int_0^{2\pi/\omega} \frac{K}{2}\,(x)^2\,\mathrm{d}t, \tag{35}$$

or, using Equations (11) and (26),

$$\mathrm{PE} = \int_0^{2\pi} \frac{K}{2}\,(Y_0(\tau))^2\,\mathrm{d}\tau. \tag{36}$$

The main thrust of this paper focuses on the influence of fractional hysteresis models on the primary amplitude frequency behavior. This is achieved by suppressing the sub/super and combination harmonic behavior. The subharmonics are removed through the use of either of the integral constraints (35) or (36). Specifically, due to their form, the orthogonality properties of harmonic functions cancel the subharmonics. On the other hand, since transient behavior is attenuated by hysteresis, then the superharmonic responses which are induced by transient inputs are eliminated by disregarding the initial conditions [15].

Expanding Y_0 in perturbation series yields the following hierarchy of constraints, namely

$$\mathrm{PE} = \frac{K}{2} \int_0^{2\pi} (Y_{00}(\tau))^2\,\mathrm{d}\tau, \tag{37}$$

$$0 = K \int_0^{2\pi} Y_{00}(\tau) Y_{01}(\tau)\, \mathrm{d}\tau, \tag{38}$$

$$0 = \frac{K}{2} \int_0^{2\pi} (2Y_{00}(\tau)Y_{02}(\tau) + (Y_{01}(\tau))^2)\, \mathrm{d}\tau, \tag{39}$$

and so on.

Based on the operational properties [14] of the derivative, the zeroth order steady-state solution takes the form

$$\underline{Y}_0 = \underline{h}_1^0 \cos(\tau) + \underline{h}_2^0 \sin(\tau), \tag{40}$$

where $\underline{h}_1^0$ and $\underline{h}_2^0$ are $(2m \times 1)$ amplitude vectors. To obtain the vectors coefficients one can substitute Equation (40) into Equation (32), and after several algebraic manipulations the following matrix equation can be found:

$$\hat{A}_{M0}(\Omega_0)\hat{\underline{h}}^0 = \hat{\underline{f}}_M, \tag{41}$$

where

$$\hat{\underline{h}}^0 = \begin{Bmatrix} \underline{h}_1^0 \\ \underline{h}_2^0 \end{Bmatrix}, \quad \hat{\underline{f}}_M = \begin{Bmatrix} \underline{f}_M \\ \underline{0} \end{Bmatrix} \tag{42}$$

are $(4m \times 1)$ vectors, and

$$\hat{A}_{M0} = \left[\begin{array}{c|c} \Omega_0 I \cos\left(\dfrac{\pi}{2m}\right) - A_{M0} & \Omega_0 I \sin\left(\dfrac{\pi}{2m}\right) \\ \hline -\Omega I \sin\left(\dfrac{\pi}{2m}\right) & \Omega_0 I \cos\left(\dfrac{\pi}{2m}\right) - A_{M0} \end{array}\right] \tag{43}$$

is an Ω_0 dependent $(4m \times 4m)$ matrix, wherein I is $(2m \times 2m)$ unity matrix. Using Equations (40) and (41) one finds that $\underline{Y}_0$ is given by

$$\underline{Y}_0 = \underline{Y}_0(\Omega_0, \tau) = [[I]\cos(\tau)[I]\sin(\tau)][\hat{A}_{M0}(\Omega_0)]^{-1}\hat{\underline{f}}_M. \tag{44}$$

To constrain the choice of Ω_0, application of Equations (37) and (40) yields the following parametric relationship between and potential energy

$$\frac{2(\mathrm{PE})}{\pi K} = (h_{01}^0(\Omega_0))^2 + (h_{02}^0(\Omega_0))^2, \tag{45}$$

where h_{01}^0 and h_{02}^0 are the zeroth terms of vectors $\underline{h}_1^0$ and $\underline{h}_2^0$, respectively. The right-hand side of Equation (45) represents the square of magnitude of the weighted potential energy. The first order set of the perturbational hierarchy is given by Equations (32) and (38).

Substituting Equation (40) into Equation (33), we get

$$\begin{aligned} \Omega_0 D_{1/m}(\underline{Y}_1 - A_{M0}\underline{Y}_1 &= -\Omega_1\{\underline{a}_1 \cos(\tau) + \underline{a}_2 \sin(\tau)\} \\ &\quad + A_{M1}\{\underline{H}_1 \cos(\tau) + \underline{H}_2 \sin(\tau) + \underline{H}_1^3 \cos(3\tau) + \underline{H}_2^3 \sin(3\tau)\}, \end{aligned} \tag{46}$$

where

$$\underline{a}_1 = \underline{h}_1^0 \cos\left(\frac{\pi}{2m}\right. + \underline{h}_2^0 \sin\left(\frac{\pi}{2m}\right. , \quad \underline{a}_2 = -\underline{h}_1^{-} \sin\left(\frac{\pi}{2m}\right. + \underline{h}_2^0 \cos\left(\frac{\pi}{2m}\right. , \tag{47}$$

$$\underline{H}_1 = \frac{3}{4}\, \underline{H} \sin(\underline{\phi}^0), \quad \underline{H}_2 = \frac{3}{4}\, \underline{H} \cos(\underline{\phi}^0), \tag{48}$$

$$\underline{H}_1^3 = -\frac{1}{4}\, \underline{H} \sin(3\underline{\phi}^0), \quad \underline{H}_2^3 = -\frac{1}{4}\, \underline{H} \cos(3\underline{\phi}^0), \tag{49}$$

and

$$\underline{H} = [(\underline{h}_1^0)^2 + (\underline{h}_2^0)^2]^{3/2}, \quad \underline{\phi}^0 = \tan^{-1}\left(\frac{\underline{h}_1^0}{\underline{h}_2^0}\right). \tag{50}$$

The solution of Equation (46) takes the following form:

$$\underline{Y}_1 = \underline{h}_1^1 \cos(\tau) + \underline{h}_2^1 \sin(\tau) + \underline{h}_1^3 \cos(3\tau) + \underline{h}_2^3 \sin(3\tau), \tag{51}$$

where $\underline{h}_1^1$, $\underline{h}_2^1$, $\underline{h}_1^3$, and $\underline{h}_2^3$ are $(2m \times 1)$ amplitude vectors.

Substituting Equation (51) into Equation (46) we see that

$$\underline{\hat{h}}^1 = \underline{\hat{h}}_1^1 + \Omega_1 \underline{\hat{h}}_2^1 \tag{52}$$

and

$$\underline{\hat{h}}_1^1 = [\hat{A}_{M0}(\Omega_0)]^{-1} \underline{\Gamma}_1, \tag{53}$$

$$\underline{\hat{h}}_2^1 = [\hat{A}_{M0}(\Omega_0)]^{-1} \underline{\Gamma}_2, \tag{54}$$

$$\underline{\hat{h}}^3 = [\hat{A}_{M0}(\Omega_0)]^{-1} \underline{\Gamma}_3, \tag{55}$$

where

$$\underline{\hat{h}}^1 = \left.\begin{matrix} \underline{h}_1^1 \\ \underline{h}_2^1 \end{matrix}\right\}, \quad \underline{\hat{h}}_1^1 = \left.\begin{matrix} \underline{h}_{11}^1 \\ \underline{h}_{21}^1 \end{matrix}\right\}, \quad \underline{\hat{h}}_2^1 = \left.\begin{matrix} \underline{h}_{12}^1 \\ \underline{h}_{22}^1 \end{matrix}\right\}, \quad \underline{\hat{h}}^3 = \left.\begin{matrix} \underline{h}_1^3 \\ \underline{h}_2^3 \end{matrix}\right\}, \tag{56}$$

$$\underline{\Gamma}_1 = \left.\begin{matrix} A_{M1}\underline{H}_1 \\ A_{M1}\underline{H}_2 \end{matrix}\right\}, \quad \underline{\Gamma}_2 = -\left.\begin{matrix} \underline{a}_1 \\ \underline{a}_2 \end{matrix}\right\}, \quad \underline{\Gamma}_3 = \left.\begin{matrix} A_{M1}\underline{H}_1^3 \\ A_{M1}\underline{H}_2^3 \end{matrix}\right\}, \tag{57}$$

and

$$\hat{A}_{M0}^3(\Omega_0) = \begin{array}{c|c} (3)^{1/m}\Omega_0 I \cos\left(\frac{\pi}{2m}\right. - A_{M0} & (3)^{1/m}\Omega_0 I \sin\left(\frac{\pi}{2m}\right. \\ \hline -(3)^{1/m}\Omega I \sin\left(\frac{\pi}{2m}\right. & (3)^{1/m}\Omega_0 I \cos\left(\frac{\pi}{2m}\right. - A_{M0} \end{array} . \tag{58}$$

To determine Ω_1, we write the expressions for zeroth term of the vectors $\underline{Y}_0$ and $\underline{Y}_1$. Thus, from Equation (40) we have

$$Y_{m0} = h_{m1}^0 \cos(\tau) + h_{m2}^0 \sin(\tau) \tag{59}$$

and from Equation (51), using definitions (52) and (56), we write that

$$\begin{aligned} Y_{m1} &= h^1_{m11}\cos(\tau) + h^1_{m21}\sin(\tau) + \Omega_1(h^1_{m12}\cos(\tau) + h^1_{m22}\sin(\tau)) \\ &\quad + h^3_{m1}\cos(3\tau) + h^3_{m2}\sin(3\tau). \end{aligned} \tag{60}$$

Substituting Equations (59) and (60) into Equation (38) one obtains

$$\Omega_1 = -\frac{h^0_{m1}h^1_{m11} + h^0_{m2}h^1_{m21}}{h^0_{m1}h^1_{m12} + h^0_{m2}h^1_{m22}}\,. \tag{61}$$

The second order set of the perturbational hierarchy is given by Equations (34) and (39). Substituting Equations (40) and (51) into Equation (34), one obtains

$$\begin{aligned} \Omega_0 D_{1/m}(\underline{Y}_2) - A_{M0}\underline{Y}_2 &= -\Omega_1\{\underline{b}_1\cos(\tau) + \underline{b}_2\sin(\tau) + \underline{b}^3_1\cos(3\tau) + \underline{b}^3_2\sin(3\tau)\} \\ &\quad - \Omega_2\{\underline{a}_1\cos(\tau) + \underline{a}_2\sin(\tau)\} \\ &\quad + \frac{3}{4}A_{M1}\{\underline{G}_1\cos(\tau) + \underline{G}_2\sin(\tau) + \underline{G}^3_1\cos(3\tau) \\ &\quad + \underline{G}^3_2\sin(3\tau) + \underline{G}^5_1\cos(5\tau) + \underline{G}^5_2\sin(5\tau)\}, \end{aligned} \tag{62}$$

where

$$\underline{b}_1 = \underline{h}^1_1\cos\left(\frac{\pi}{2m}\right) + \underline{h}^1_2\sin\left(\frac{\pi}{2m}\right), \quad \underline{b}_2 = -\underline{h}^1_1\sin\left(\frac{\pi}{2m}\right) + \underline{h}^1_2\cos\left(\frac{\pi}{2m}\right), \tag{63}$$

$$\begin{aligned} \underline{b}^3_1 &= (3)^{1/m}\underline{h}^3_1\cos\left(\frac{\pi}{2m}\right) + (3)^{1/m}\underline{h}^3_2\sin\left(\frac{\pi}{2m}\right), \\ \underline{b}^3_2 &= -(3)^{1/m}\underline{h}^3_1\sin\left(\frac{\pi}{2m}\right) + (3)^{1/m}\underline{h}^3_2\cos\left(\frac{\pi}{2m}\right), \end{aligned} \tag{64}$$

$$\begin{aligned} \underline{G}_1 &= (\underline{h}^0_1)^2(3\underline{h}^1_1 + \underline{h}^3_1) + 2\underline{h}^0_1\underline{h}^0_2(\underline{h}^1_2 + \underline{h}^3_2) + (\underline{h}^0_2)^2(\underline{h}^1_1 - \underline{h}^3_1), \\ \underline{G}_2 &= (\underline{h}^0_1)^2(\underline{h}^1_2 + \underline{h}^3_2) + 2\underline{h}^0_1\underline{h}^0_2(\underline{h}^1_1 - \underline{h}^3_1) + (\underline{h}^0_2)^2(3\underline{h}^1_2 - \underline{h}^3_2), \end{aligned} \tag{65}$$

$$\begin{aligned} \underline{G}^3_1 &= (\underline{h}^0_1)^2(\underline{h}^1_1 + 2\underline{h}^3_1) - 2\underline{h}^0_1\underline{h}^0_2\underline{h}^1_2 + (\underline{h}^0_2)^2(-\underline{h}^1_1 + 2\underline{h}^3_1), \\ \underline{G}^3_2 &= (\underline{h}^0_1)^2(\underline{h}^1_2 + 2\underline{h}^3_2) + 2\underline{h}^0_1\underline{h}^0_2\underline{h}^1_1 + (\underline{h}^0_2)^2(-\underline{h}^1_2 + 2\underline{h}^3_2), \end{aligned} \tag{66}$$

$$\begin{aligned} \underline{G}^5_1 &= \underline{h}^3_1(\underline{h}^0_1)^2 - 2\underline{h}^0_1\underline{h}^0_2\underline{h}^3_2 - \underline{h}^3_1(\underline{h}^0_2)^2, \\ \underline{G}^5_2 &= \underline{h}^3_2(\underline{h}^0_1)^2 + 2\underline{h}^0_1\underline{h}^0_2\underline{h}^3_1 - \underline{h}^3_2(\underline{h}^0_2)^2. \end{aligned} \tag{67}$$

The solution form of Equation (62) takes the following format

$$\underline{Y}_2 = \underline{g}^1_1\cos(\tau) + \underline{g}^1_2\sin(\tau) + \underline{g}^3_1\cos(3\tau) + \underline{g}^3_2\sin(3\tau) + \underline{g}^5_1\cos(5\tau) + \underline{g}^5_2\sin(5\tau), \tag{68}$$

where $\underline{g}^1_1$, $\underline{g}^1_2$, $\underline{g}^3_1$, $\underline{g}^3_2$ $\underline{g}^5_1$, and $\underline{g}^5_2$ are $(2m \times 1)$ amplitude vectors.

Substituting Equation (68) into Equation (62) we see that

$$\begin{aligned} \underline{\hat{g}}^1 &= \underline{\hat{g}}^1_1 + \Omega_1\underline{\hat{g}}^1_2 + \Omega_2\underline{\hat{g}}^1_3, \\ \underline{\hat{g}}^3 &= \underline{\hat{g}}^3_1 + \Omega_1\underline{\hat{g}}^3_2, \end{aligned} \tag{69}$$

and

$$\underline{\hat{g}}_1^1 = [\hat{A}_{M0}(\Omega_0)]^{-1}\underline{\Psi}_1, \tag{70}$$

$$\underline{\hat{g}}_2^1 = [\hat{A}_{M0}(\Omega_0)]^{-1}\underline{\Psi}_2, \tag{71}$$

$$\underline{\hat{g}}_3^1 = [\hat{A}_{M0}(\Omega_0)]^{-1}\underline{\Gamma}_2, \tag{72}$$

$$\underline{\hat{g}}_1^3 = [\hat{A}_{M0}^3(\Omega_0)]^{-1}\underline{\Psi}_3, \tag{73}$$

$$\underline{\hat{g}}_2^3 = [\hat{A}_{M0}^3(\Omega_0)]^{-1}\underline{\Psi}_4, \tag{74}$$

$$\underline{\hat{g}}^5 = [\hat{A}_{M0}^5(\Omega_0)]^{-1}\underline{\Psi}_5, \tag{75}$$

where

$$\underline{\hat{g}}^1 = \begin{matrix} \underline{g}_1^1 \\ \underline{g}_2^1 \end{matrix} \quad , \quad \underline{\hat{g}}_1^1 = \begin{matrix} \underline{g}_{11}^1 \\ \underline{g}_{21}^1 \end{matrix} \quad , \quad \underline{\hat{g}}_2^1 = \begin{matrix} \underline{g}_{12}^1 \\ \underline{g}_{22}^1 \end{matrix} \quad , \quad \underline{\hat{g}}_3^1 = \begin{matrix} \underline{g}_{13}^1 \\ \underline{g}_{23}^1 \end{matrix} \quad , \tag{76}$$

$$\underline{\hat{g}}^3 = \begin{matrix} \underline{g}_1^3 \\ \underline{g}_2^3 \end{matrix} \quad , \quad \underline{\hat{g}}_1^3 = \begin{matrix} \underline{g}_{11}^3 \\ \underline{g}_{21}^3 \end{matrix} \quad , \quad \underline{\hat{g}}_2^3 = \begin{matrix} \underline{g}_{12}^3 \\ \underline{g}_{22}^3 \end{matrix} \quad , \quad \underline{\hat{g}}^5 = \begin{matrix} \underline{g}_1^5 \\ \underline{g}_2^5 \end{matrix} \quad , \tag{77}$$

and

$$\begin{aligned} \underline{\Psi}_1 &= \frac{3}{4} \begin{matrix} A_{M1}\underline{G}_1 \\ A_{M1}\underline{G}_2 \end{matrix} \quad , \quad \underline{\Psi}_2 = - \begin{matrix} \underline{b}_1 \\ \underline{b}_2 \end{matrix} \quad , \quad \underline{\Psi}_3 = \frac{3}{4} \begin{matrix} A_{M1}\underline{G}_1^3 \\ A_{M1}\underline{G}_2^3 \end{matrix} \quad , \\ \underline{\Psi}_4 &= - \begin{matrix} \underline{b}_1^3 \\ \underline{b}_2^3 \end{matrix} \quad , \quad \underline{\Psi}_5 = \frac{3}{4} \begin{matrix} A_{M1}\underline{G}_1^5 \\ A_{M1}\underline{G}_2^5 \end{matrix} \end{aligned} \tag{78}$$

and

$$\hat{A}_{M0}^5(\Omega_0) = \begin{array}{c|c} (5)^{1/m}\Omega_0 I \cos\left(\frac{\pi}{2m} - A_{M0}\right. & (5)^{1/m}\Omega_0 I \sin\left(\frac{\pi}{2m}\right. \\ \hline -(5)^{1/m}\Omega I \sin\left(\frac{\pi}{2m}\right. & (5)^{1/m}\Omega_0 I \cos\left(\frac{\pi}{2m} - A_{M0}\right. \end{array} \quad . \tag{79}$$

To determine Ω_2 we write the expressions for zeroth term of the vector $\underline{Y}_2$, e.g.,

$$\begin{aligned} Y_{m2} \;=\; & g_{m11}^1 \cos(\tau) + g_{m21}^1 \sin(\tau) + \Omega_1(g_{m12}^1 \cos(\tau) + g_{m22}^1 \sin(\tau)) \\ & + \Omega_2(g_{m13}^1 \cos(\tau) + g_{m23}^1 \sin(\tau)) + g_{m11}^3 \cos(3\tau) + g_{m21}^3 \sin(3\tau) \\ & + \Omega_1(g_{m12}^3 \cos(3\tau) + g_{m22}^3 \sin(3\tau)) + g_{m1}^5 \cos(5\tau) + g_{m2}^5 \sin(5\tau). \end{aligned} \tag{80}$$

Substituting Equations (59), (60), and (80) into Equation (39) one obtains

$$\begin{aligned} \Omega_2 \;=\; & -\frac{(h_{011}^1)^2 + (h_{021}^1)^2 + (h_{01}^3)^2 + (h_{02}^3)^2 + 2(g_{011}^1 h_{01}^1 + g_{021}^1 h_{02}^0)}{2(h_{01}^0 g_{013}^1 + h_{02}^0 g_{023}^1)} \\ & \times \; \frac{2(h_{011}^1 h_{012}^1 + h_{021}^1 h_{022}^1 + g_{012}^1 h_{01}^0 + g_{022}^1 h_{02}^0)\Omega_1 + ((h_{012}^1)^2 + (h_{022}^1)^2)\Omega_1^2}{2(h_{01}^0 g_{013}^1 + h_{02}^0 g_{023}^1)} \; . \end{aligned} \tag{81}$$

Thus, the total solution of Equation (1) becomes

$$\underline{Y} = \underline{Y}_0(\Omega_0, \tau) + \varepsilon \underline{Y}_1(\Omega_0, \Omega_1(\Omega_0), \tau) + \varepsilon^3 \underline{Y}_2(\Omega_0, \Omega_1(\Omega_0), \Omega_2(\Omega_0, \Omega_1), \tau) + O(\varepsilon^3) \quad (82)$$

with

$$\Omega = \Omega_0 + \varepsilon \Omega_1(\Omega_0) + \varepsilon^2 \Omega_2(\Omega_0, \Omega_1) + O(\varepsilon^3). \quad (83)$$

Since $\tau = \omega t$ and $\omega^{1/m} = \Omega$, therefore we have

$$\omega = [\Omega_0 + \varepsilon \Omega_1(\Omega_0) + \varepsilon^2 (\Omega_2)(\Omega_0, \Omega_1) + O(\varepsilon^3)]^m. \quad (84)$$

Equations (82) and (83) represent the overall solution in Ω_0 parametric space. In particular, since all Ω_i are defined in terms of Ω_0, it follows that $\underline{Y} = \underline{Y}(\Omega_0, \tau)$ and $\omega = \omega(\Omega_0)$. Hence, to establish the relationship $\underline{Y} = \underline{Y}(\omega, \tau)$, $\underline{Y}(\Omega_0, t)$ and $\omega(\Omega_0)$ are cross referenced along the entire span of the Ω_0 range space. This process will be illustrated in the next section.

4. Discussion

In the preceding sections, a general steady-state solution was developed for Duffing's equation with hysteresis defined by diophantinized fractional representations. Of particular interest is a determination of the effects of fractional order on the full range of amplitude-frequency behavior. To achieve this, a single operator hysteretic formulation will be considered, namely

$$M D_2(x) + C D_q(x) + Kx + \mu(x)^3 = f \cos(\omega t). \quad (85)$$

To illustrate the solution in parametric space, for $q = 1$, Figures 1a and 1b depict the functional properties of Ω and x in Ω_0 parametric space. As noted earlier, to determine the $x = x(\omega, t)$ relationship, $x(\Omega_0, t)$ and $\Omega(\Omega_0)$ must be cross referenced over the full span of the $\Omega_0$range space. The result of this operation is given in Figure 1c wherein the damped nonlinear frequency response is given.

As q is varied, the response behavior is modulated both in frequency and amplitude. This is clearly illustrated in Figures 2 and 3. The shifts can be explained by the direction in which $D_q(\cdot)$ changed. For instance, as $D_q(\cdot)$ tends towards $D_0(\cdot)$, combined dissipative and spring like behavior is excited. Here as the effective stiffness grows, the apparent system frequency increases leading to a rightward shift of the amplitude frequency response, as is illustrated in Figure 2. The increase in amplitude is a result of the reduced dissipation. When $D_q(\cdot)$ tends towards $D_2(\cdot)$, both dissipative and mass like effects are expected. As the pseudo-mass grows the apparent system frequency reduces leading to a leftward (downward) shift in the frequency response (see Figure 3). Concomitantly, the amplitude peak again rises as a result of the reduced dissipation.

As q is varied over wider ranges, Figures 4 and 5 illustrate the leftward and rightward biasing of the peak of the amplitude response behavior. Again amplitude grows as q is shifted away from the $q = 1$, e.g., the pure dissipation mode. The effect of fractional order on the phase angle for the case before ($\Omega_0 = 0.5$) and after ($\Omega_0 = 1.5$) resonance is shown in Figures 6 and 7, respectively.

In the context of foregoing, it follows that as q is allowed to range, the operator can either represent combined dissipative-kinetic or dissipative-stiffness behavior. When several operators are combined in a simulation, e.g., as in Equation (1), a very rich assortment of

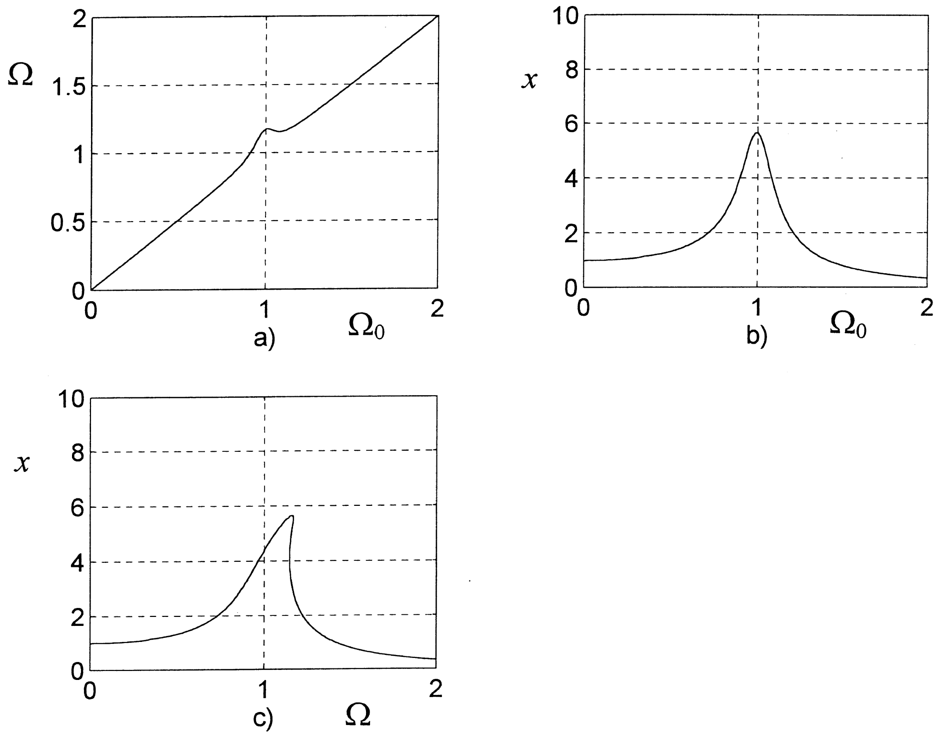

Figure 1. The (Ω, Ω_0), (x, Ω_0) and (x, Ω) relationships for $\ddot{x} + 0.15D_1(x) + x + 0.01x^3 = \cos(\omega t)$.

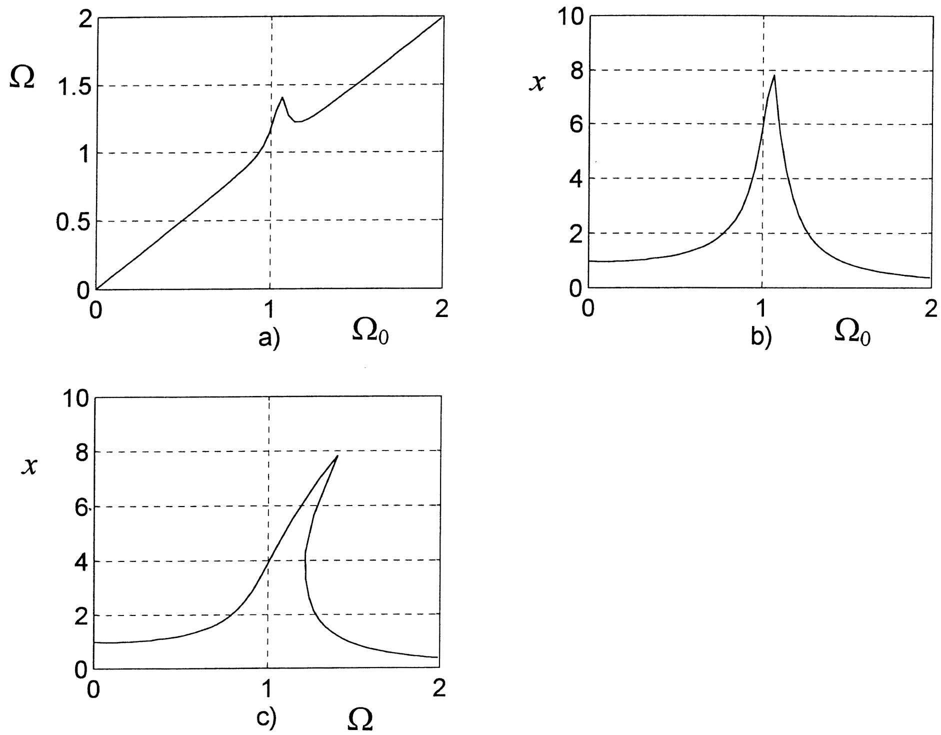

Figure 2. The (Ω, Ω_0), (x, Ω_0) and (x, Ω) relationships for $\ddot{x} + 0.15D_{1/2}(x) + x + 0.01x^3 = \cos(\omega t)$.

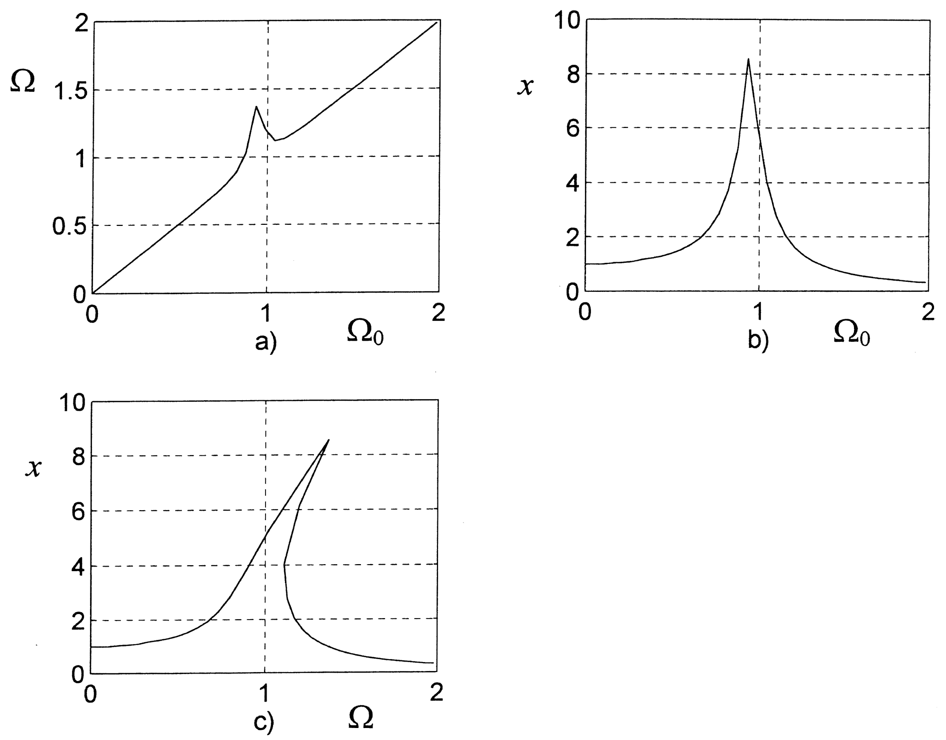

Figure 3. The (Ω, Ω_0), (x, Ω_0) and (x, Ω) relationships for $\ddot{x} + 0.15D_{3/2}(x) + x + 0.01x^3 = \cos(\omega t)$.

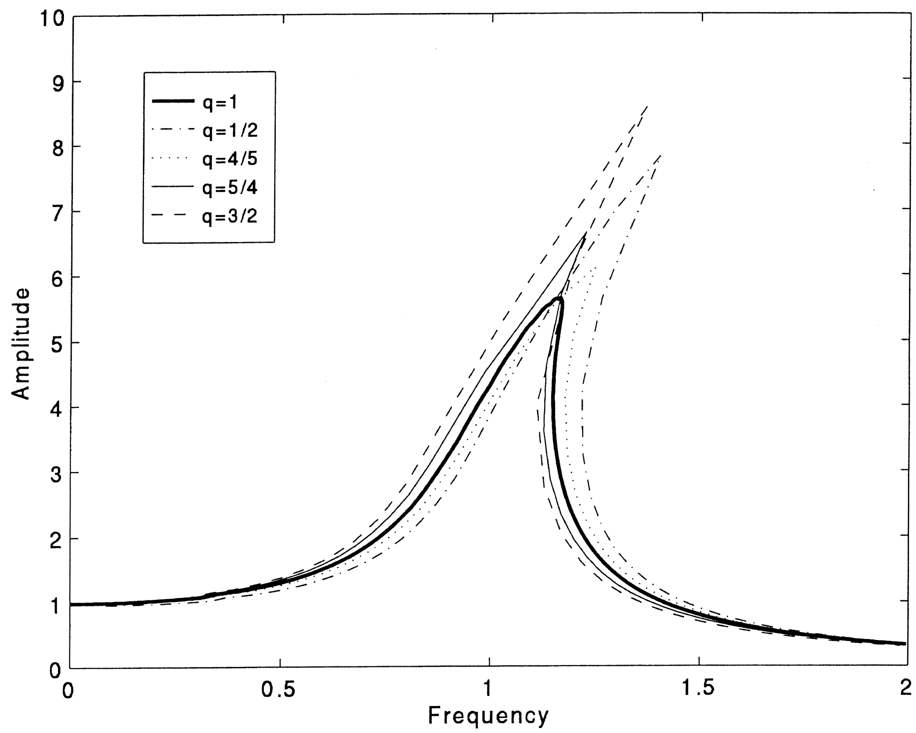

Figure 4. Effect of fractional damping on the response of the Duffing equation.

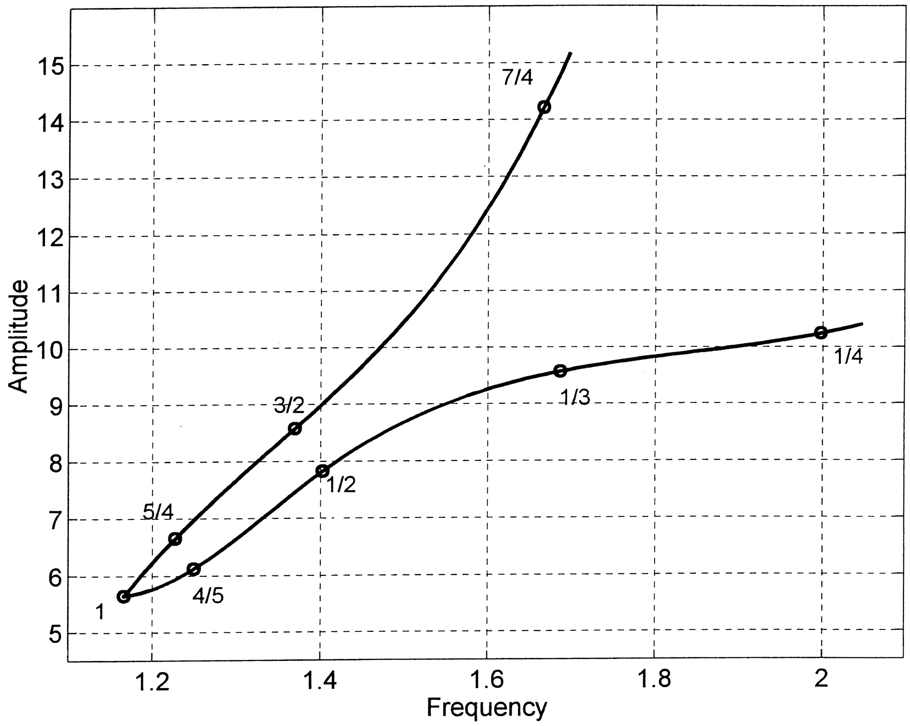

Figure 5. Locus of the amplitude maxima for the Duffing equation response curves.

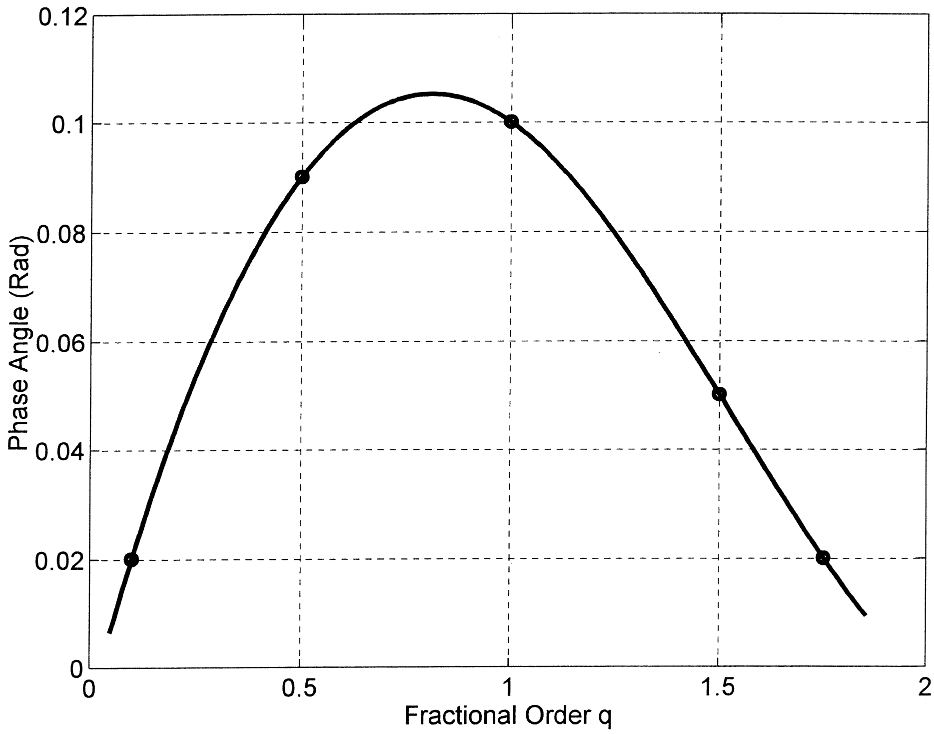

Figure 6. Phase angle *versus* fractional order in the region before the resonance ($\Omega_0 = 0.5$).

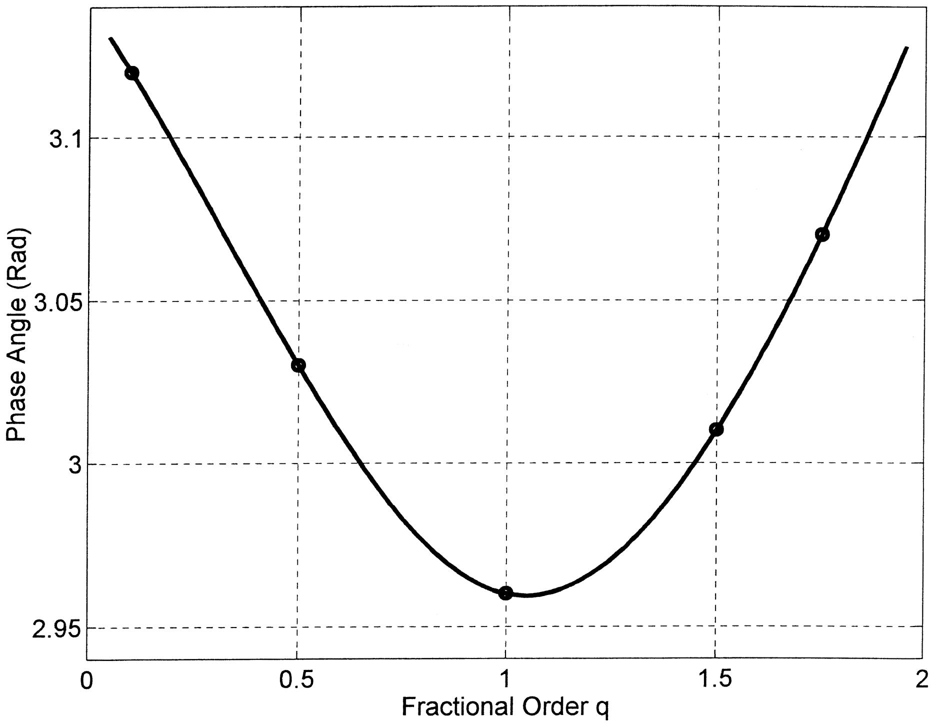

Figure 7. Phase angle *versus* fractional order in the region after the resonance ($\Omega_0 = 1.5$).

response characteristics can be simulated. Note that the generality of the current solution is such that both fractional derivatives as well as integrals can be represented, e.g., $q > 0$ and $q < 0$, respectively. Further, because of the vector formulation of the governing equations, the results can be directly extended to large scale systems with general hysteresis modeled by differ-integro operators.

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